Saturs

1. Bra-Ket Notation

1.1. Ket $|\psi\rangle$

Represents a column vector for a quantum state.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \iff \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

1.1.1. Basis states

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

1.2. Bra $\langle\psi|$

Represents a **conjugate transpose vector (kompleksi saistīts)** (row vector) of $|\psi\rangle$.

$$\text{If } \ |\psi\rangle = \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \text{ , then } |\psi\rangle = \begin{pmatrix}a^* & b^*\end{pmatrix}$$

1.3. Scalar Product $\langle\varphi|\psi\rangle$

Inner product of two states.

$$\text{If } \ |\varphi\rangle = \gamma|0\rangle + \delta|1\rangle \text{ , then } \langle\varphi|\psi\rangle = \gamma^*\alpha + \delta^*\beta$$

1.3.1. Orthogonal states

$$\langle\varphi|\psi\rangle = 0$$

1.4. Projection $\langle i|\psi\rangle$

Amplitude of the basis state $|i\rangle$ in $|\psi\rangle$.

For $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle : \langle 0|\psi\rangle = \alpha, \psi\langle 1|\psi\rangle = \beta.$

Probability of measuring state $|i\rangle : P(i) = |\langle i|\psi\rangle|^2$

2. Fundamentals

2.1. Qubit (Kvantu bits)

2.1.1. Basis states

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

2.1.2. Superposition

A qubit can be in a linear combination of basis states:

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \text{ , where } \alpha, \beta \in \mathbb{C}$$

are probability amplitudes.

2.1.3. Normalization

$$|\alpha|^2 + |\beta|^2 = 1$$

$|\alpha|^2$ is the probability of measuring $|0\rangle$, $|\beta|^2$ is the probability of measuring $|1\rangle$.

2.1.4. Bloch Sphere

Geometric representation of a single qubit state:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$$

2.2. Measurement (Mērījumi)

- Projective measurement in the basis (e.g. computational $\{|0\rangle, |1\rangle\}$ or Hadamard $\{|+\rangle, |-\rangle\}$).
- If state is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$:
  - Outcome 0: probability $P(0) = |\langle 0|\psi\rangle|^2 = |\alpha|^2$. Post-measurement state: $|0\rangle$.
  - Outcome 1: probability $P(1) = |\langle 1|\psi\rangle|^2 = |\beta|^2$. Post-measurement state: $|1\rangle$.
- Measurement collapses the superposition (mērījums maina kvantu bitu (observer effect)).

2.2.1. Measurement operators

$$M_0 = |0\rangle\langle 0|, M_1 = |1\rangle\langle 1|$$

$$\sum_m M_m^\dagger M_m = I$$

2.2.2. Measuring in $|+\rangle, |-\rangle$ basis

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

To measure $|0\rangle$ in this basis: $|0\rangle = \frac{1}{\sqrt{+}} + \frac{1}{\sqrt{2}}|-\rangle$.

$$P(+) = |\langle +|0\rangle|^2 = \frac{1}{2}, P(-) = |\langle -|1\rangle|^2 = \frac{1}{2}$$

2.2.2.1. Example: $|\psi\rangle = \frac{1+2i}{\sqrt{7}}|0\rangle + \frac{1-i}{\sqrt{7}}|1\rangle$

$$P(0) = \left|\frac{1+2i}{\sqrt{7}}\right|^2 = \frac{1^2+2^2}{7} = \frac{5}{7}$$

$$P(1) = \left|\frac{1-i}{\sqrt{7}}\right|^2 = \frac{1^2+(-1)^2}{7} = \frac{2}{7}$$

> 🔥 **Tip**
>
> Sum must be 1.

3. Single Qubit Unitary Transformations

Quantum gates are unitary matrices U.

- Unitary condition: $UU^\dagger = U^\dagger U = I$, where $U^\dagger$ is the conjugate transpose.
- Action on state $|\psi'\rangle = U|\psi\rangle$

3.0.1. Properties

Linearity $(U(\alpha|\psi_1\rangle + \beta|\psi_2\rangle) = \alpha U|\psi_1\rangle + \beta U|\psi_2\rangle)$ and preserves vector length.

3.0.2. Matix form

If $U|0\rangle = a|0\rangle + b|1\rangle)$ and $U|1\rangle = c|0\rangle + d|1\rangle$, then

$$U = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Columns (and rows) must be orthonormal vectors: $\vec{v_1^*} \cdot \vec{v_2} = 0$ and $|\vec{v_1}|^2 = 1$.

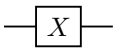
3.1. Pauli Gates

3.1.1. I (Identity)

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{cases} I|0\rangle = |0\rangle \\ I|1\rangle = |1\rangle \end{cases}$$

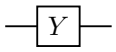
3.1.2. X (NOT)

Bit flip



$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{cases} X|0\rangle = |1\rangle \\ X|1\rangle = |0\rangle \end{cases}$$

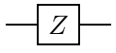
3.1.3. Y Gate



$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \begin{cases} Y|0\rangle = -i|1\rangle \\ Y|1\rangle = i|0\rangle \end{cases}$$

3.1.4. Z Gate

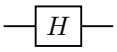
Phase flip



$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \begin{cases} Z|0\rangle = |0\rangle \\ Z|1\rangle = -|1\rangle \end{cases}$$

3.2. Hadamard Gate (H)

Creates superpositions

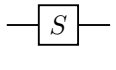


$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \begin{cases} H|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \\ H|-1\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \end{cases}$$

$$\begin{cases} H|0\rangle = |+\rangle \\ H|1\rangle = |-\rangle \end{cases} \qquad HH = H^2 = I$$

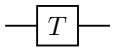
3.3. Phase Gates

3.3.1. S Gate $\left(\sqrt{Z}\right)$



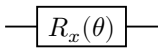
$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad S^2 = Z$$

3.3.2. T Gate $\left(\frac{\pi}{8}\right)$

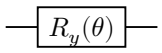


$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} \qquad T^2 = S$$

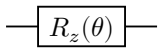
3.4. Rotation Gates ($R_n(\theta)$)



$$R_x(\theta) = e^{\frac{-i\theta X}{2}} = \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$



$$R_y(\theta) = e^{\frac{-i\theta Y}{2}} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}$$



$$R_z(\theta) = e^{\frac{-i\theta Z}{2}} = \begin{pmatrix} e^{\frac{-i\theta}{2}} & 00 \\ & e^{\frac{i\theta}{2}} \end{pmatrix}$$

> 🔥 **Tip**
>
> $R_\alpha : \begin{cases} R_\alpha|0\rangle : \cos\alpha|0\rangle + \sin\alpha|1\rangle \\ R_\alpha|1\rangle : -\sin\alpha|0\rangle + \cos\alpha|1\rangle \end{cases}$. This is $R_{y(-2\alpha)}$.

3.5. Game Compositions

Applied right to left. $UV|\psi\rangle = U(V|\psi\rangle)$.

- $HZH = X$
- $HXH = Z$

3.6. Inverse Tranformation

$$U^{-1} = U^\dagger$$

3.7. Non-Unitary Operations

(Not physically realizable as closed system evolution)

3.7.1. Qubit Deletion

$$\begin{cases} U|0\rangle = |0\rangle \\ U|1\rangle = |0\rangle \end{cases}$$

4. Multi-Qubit Systems

4.1. Tensor product

Combines state space.

$$(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle) =$$

$$= ac|00\rangle + ad|01\rangle + bc|10\rangle + bd|11\rangle$$

For k qubits, 2^k basis states.

4.1.1. Operators

$$(A \otimes B)(|\psi_A\rangle \otimes |\psi_B\rangle) = (A|\psi_A\rangle) \otimes (B|\psi_B\rangle)$$

4.2. Product States vs. Entangled States

4.2.1. Product state

Can be written as $|\psi_A\rangle \otimes |\psi_B\rangle$.

4.2.1.1. Example

$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) =$$

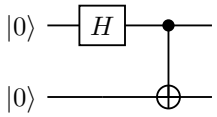
$$= \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right) \otimes \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)$$

4.2.2. Entangled States

Cannot be factored

4.2.2.1. Example

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) (\text{Bell state } |\Phi^+\rangle)$$



4.3. Multi-Qubit Measurement & Normalization

Measure one qubit from a multi-qubit system.

4.3.0.1. Example

4.3.0.1.1. State

$$a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$

4.3.0.1.2. Measure 1st qubit

4.3.0.1.2.1. Prob of 0

$$P(q_1 = 0) = |a|^2 + |b|^2$$

Post-measurement state:

$$\frac{a|00\rangle + b|01\rangle}{\sqrt{|a|^2 + |b|^2}} = |0\rangle \otimes \frac{a|0\rangle + b|1\rangle}{\sqrt{|a|^2 + |b|^2}}$$

4.3.0.1.2.2. Prob of 1

$$P(q_1 = 1) = |c|^2 + |d|^2$$

Post-measurement state:

$$\frac{c|10\rangle + d|11\rangle}{\sqrt{|c|^2 + |d|^2}} = |1\rangle \otimes \frac{c|0\rangle + d|1\rangle}{\sqrt{|c|^2 + |d|^2}}$$

4.3.0.2. Example

4.3.0.2.1. State

$$\frac{2}{3}|00\rangle + \frac{1}{3}|01\rangle + \frac{2}{3}|10\rangle$$

4.3.0.2.2. Measure 1st qubit

4.3.0.2.2.1. Prob of 0

$$P(0) = \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{5}{9}$$

State of 2nd qubit:

$$\frac{\frac{2}{3}|0\rangle + \frac{1}{3}|1\rangle}{\sqrt{\frac{5}{9}}} = \frac{1}{\sqrt{5}}(2|0\rangle + |1\rangle)$$

4.3.0.2.2.2. Prob of 1

$$P(1) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

State of 2nd qubit:

$$\frac{\frac{2}{3}|0\rangle}{\sqrt{\frac{4}{9}}} = |0\rangle$$

5. Quantum Logic Gates & Circuits

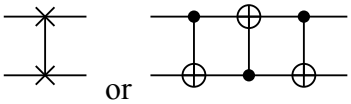
5.1. Common Multi-Qubit Gates

5.1.1. $CNOT$ (Controlled-NOT)

$$CNOT|c\rangle|t\rangle = |c\rangle|t \otimes c\rangle$$

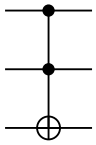
5.1.2. $SWAP$ Gate

Swaps two qubits.



5.1.3. Toffoli Gate ($CCNOT$)

$$CCNOT|c_1c_2t\rangle = |c_1c_2t \otimes (c_1 \cdot c_2)\rangle$$



5.1.4. Universality (Universalitāte)

Jebkurai unitārai U: $U = U_m U_{m-1} ... U_1$, katra U_i maina tikai 2 bāzes stāvokļus

$$U_i|x_1...x_n\rangle = a|x_1...x_n\rangle + b|y_1...y_n\rangle$$

$$U_i|y_1...y_n\rangle = c|x_1...x_n\rangle + d|y_1...y_n\rangle$$

$$U_i|z_1...z_n\rangle = |z_1...z_n\rangle$$

ja $z_1...z_n \neq x_1...x_n$, $z_1...z_n \neq y_1..y_n$.

6. Key Quantum Protocols & Concepts

6.1. No-Cloning Theorem

Impossible to create and identical copy of an arbitrary unknown quantum state.

6.2. Quantum Teleprotation

Transmits $|\psi\rangle = a|0\rangle + b|1\rangle$ using an entangled pair $|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ and 2 classical bits.

Initial state (Alice has $|\psi\rangle_C$ and qubit A, Bob has B):

$$|\psi\rangle_C \otimes |\Phi^+\rangle_{AB} =$$

$$= (a|0\rangle_C + b|1\rangle_C)\frac{1}{\sqrt{2}}(|0_A 0_B\rangle + |1_A 1_B\rangle) =$$

$$= \frac{a}{\sqrt{2}}|000\rangle + \frac{a}{\sqrt{2}}|011\rangle + \frac{b}{\sqrt{2}}|100\rangle + \frac{b}{\sqrt{2}}$$

(qubits C, A, B)

Alice applied $CNOT$ (C is control, A is target):

$$\frac{a}{\sqrt{2}}|000\rangle + \frac{a}{\sqrt{2}}|011\rangle + \frac{b}{\sqrt{2}}|110\rangle + \frac{b}{\sqrt{2}}|101\rangle$$

Alice applies H to qubit C:

$$\frac{1}{2}[a(|0\rangle + |1\rangle)|11\rangle + a(|0\rangle + |1\rangle)|11\rangle +$$

$$+b(|0\rangle - |1\rangle)|10\rangle + b(|0\rangle - |1\rangle)|01\rangle]$$

Regroup by Alice's qubits CA:

$$\frac{1}{2}[|00\rangle_{CA}(a|0\rangle + b|1\rangle) + |01\rangle_{CA}(a|1\rangle + b|0\rangle) +$$

$$+|10\rangle_{CA}(a|0\rangle - b|1\rangle) + |11\rangle_{CA}(a|1\rangle - b|0\rangle)]$$

Alice measures CA, sends 2 classical bits to Bob. Bob applies correction to his qubit B:

- Alice gets 00 $\Longrightarrow$ Bob has $a|0\rangle + b|1\rangle$ (Needs I).
- Alice gets 01 $\Longrightarrow$ Bob has $a|1\rangle + b|0\rangle$ (Needs X).
- Alice gets 10 $\Longrightarrow$ Bob has $a|0\rangle - b|1\rangle$ (Needs Z).
- Alice gets 11 $\Longrightarrow$ Bob has $a|1\rangle - b|0\rangle$ (Needs ZX).

6.3. Dense Coding (Bļīvā kodēšana)

Sends 2 classical bits of information From Alice to Bob by sending only 1 qubit, using pre-shared entangled pair.

6.3.1. Steps

1) Alice and Bob share $|\Phi^+\rangle_{AB}$
2) To send classical bits xy:
   - 00: Alice does nothing (applies I) to her qubit.
   - 01: Alice applies X to her qubit.
   - 10: Alice applies Z to her qubit.
   - 11: Alice applies X then Z (or iY) to her qubit.
3) Alice sends her modified qubit to Bob.
4) Bob performs a Bell measurement on the two qubits he now possesses to recover xy.

7. Quantum Algorithms

7.1. Oracle (U_f)

Black box for $f(x)$.

$$U_f|x\rangle|y\rangle = |x\rangle|y \otimes f(x)\rangle$$

7.1.1. Phase kickback

$$\text{If } \ |y\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \text{then}$$

$$U_f|x\rangle|-\rangle = (-1)^{f(x)}|x\rangle|-\rangle$$

7.2. Deutsch's Algorithm

- Problem: $f : \{0,1\} \to \{0,1\}$ Constant or balanced? One query.
- Task: Calculate $(f(0) + f(1))(\mathrm{mod}\, 2)$.
- Circuit:

$$|0\rangle|1\rangle \xrightarrow{H\otimes H} \frac{1}{2}\sum_x |x\rangle(|0\rangle - |1\rangle) \xrightarrow{U_f}$$

$$\xrightarrow{U_f} \frac{1}{2}\sum_x (-1)^{f(x)}|x\rangle(|0\rangle - |1\rangle) \xrightarrow{H\otimes I}$$

$$\xrightarrow[\text{Result on 1st qubit:}]{H\otimes I} \frac{1}{2}\big((-1)^{f(0)} + (-1)^{f(1)}\big)|0\rangle +$$

$$+\frac{1}{2}\big((-1)^{f(0)} - (-1)^{f(1)}\big)|1\rangle$$

- Measure 1st qubit:
  - If $f(0) = f(1)$ (constant), state is $\pm|0\rangle$. Measure 0. $(f(0) + f(1))(\mathrm{mod}\, 2) = 0$.
  - If $f(0) \neq f(1)$ (balanced), state is $\pm|1\rangle$. Measure 1. $(f(0) + f(1))(\mathrm{mod}\, 2) = 1$.

7.3. Grover's Search Algorithm

- Problem: Find x_w s.t. $f(x_w) = 1$ (marked item) in $N = 2^n$ items.
- Oracle O: $O|x\rangle = (-1)^{f(x)}|x\rangle$.
- Grover Diffusion Operator D (Inversion about the mean): $D = 2|s\rangle\langle s| - I$, where $|s\rangle = H^{\otimes n}|0\rangle^{\otimes n}$.

- Grover Iteration: $G = DO$.
- Algorithm:
  1) Start $|s\rangle$.
  2) Repeat G for k iterations.
     - 1 marked item ($L = 1$): $k \approx \frac{\pi}{4}\sqrt{N}$
     - L marked items: $k \approx \frac{\pi}{4}\sqrt{\frac{N}{L}}$
  3) Measure. High probability of marked item.
- Geometric Interpretation: Rotation in 2D plane spanned by $|s\rangle$ and $|w\rangle$ (superposition of marked items).

  More precisely, plane spanned by $|\Psi_1\rangle = \frac{1}{\sqrt{L}}\sum_{x:f(x)=1}|x\rangle$ and $|\Psi_0\rangle = \frac{1}{\sqrt{N-L}}\sum_{x:f(x)-0}|x\rangle$.
  Initial state $|s\rangle = \sin\alpha|\Psi_1\rangle + \cos\alpha|\Psi_0\rangle$, where $\sin\alpha = \sqrt{\frac{L}{N}}$.

  Oracle O reflects about $|\Psi_0\rangle$. Diffusion D reflects about $|s\rangle$. Each Iteration $G = DO$ rotates by 2α.

  After k iterations, angle with $|\Psi_0\rangle$ is $(2k+1)\alpha$.

  Prob. of measuring a marked item: $\sin^2((2k+1)\alpha)$.
- Unknown L: Iterative deepening: try iterations $t = 1, 3, 3^2, ..., \text{up to} \approx \sqrt{N}$

7.4. Quantum Fourier Transform (QFT)

- Definition:

$$QFT_N|j\rangle = \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{\frac{2\pi ijk}{N}}|k\rangle$$

  Let $\omega_N = e^{\frac{2\pi i}{N}}$.
- Circuit: Uses H and controlled-R_m gates (

$$R_m = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi i}{2^m}} \end{pmatrix}$$

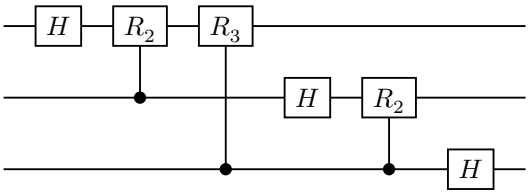
  ), then $SWAP$s.
- Property (Periodicity): If input is periodic sum

$$|\psi\rangle = \frac{1}{\sqrt{q}}\sum_{l=0}^{q-1}|a + lp\rangle$$

  (where $N = pq$, period p), then

$$QFT_N|\psi\rangle = \frac{1}{\sqrt{p}}\sum_{k=0}^{p-1} c_k\left|k\cdot\frac{N}{p}\right\rangle$$

  (Output is superposition of multiples of $\frac{N}{p}$).



7.5. Period Finding (Kvantu algoritms perioda atrašanai)

- Problem: Given $f(x) = f(x + r)$, find period r. N is size of domain.
- Algorithm:
  1) $$\frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle|0\rangle \xrightarrow[f]{U} \frac{1}{\sqrt{N}}\sum_{x=0}^{N-1}|x\rangle|f(x)\rangle$$
  2) Measure 2nd register (gets some y_0). 1st register becomes

$$\frac{1}{\sqrt{M}}\sum_{k:f(k)=y_0}|k\rangle \approx \frac{1}{\sqrt{\frac{N}{r}}}\sum_{j=0}^{\frac{N}{r}-1}|x_0 + jr\rangle$$

  3) Apply QFT_N to 1st register. Result is superposition of states $k \cdot \frac{N}{r}$.
  4) Measure 1st register. Get some value $m_j \approx j \cdot \frac{N}{r}$.
- Classical Post-processing: If N is a multiple of r: $m_j = j \cdot \frac{N}{r}$. Find r using $\gcd(m_j, N)$ and continued fractions / Euclidean algorithm to find $\frac{j}{r}$ from $\frac{m_j}{N}$.

7.6. Shor's Algorithm (Skaitļa sadalīšanai reizinātājos)

- Reduces factoring N to finding order r of $a^x \pmod N$. Uses Period Finding for $f(x) = a^x \pmod N$.
- Order r of $a^x \pmod N$: Smallest $r > 0$ s.t. $a^r \equiv 1 \pmod N$.

7.7. Simon's Algorithm

- Problem: Given $f : \{0,1\}^n \to \{0,1\}^n$ such that $f(x) = f(y)$ if $x \oplus y = s$ for some secret string $s \in \{0,1\}^n$ (or $x = y$, i.e., $s = 0^n$). Find s.
- Algorithm:
  1) Prepare $H^{\otimes n}|0\rangle^{\otimes n}|0\rangle^{\otimes n}$.
  2) Apply: $U_f : \frac{1}{\sqrt{2^n}}\sum_x|x\rangle|f(x)\rangle$.

3) Measure second register. First register collapses to $\frac{1}{\sqrt{2}}(|x_0\rangle + |x_0 \oplus s\rangle)$ for some x_0.
4) Apply $H^{\otimes n}$ to the first register.
5) Measure first register to get string y such that $y \cdot s = 0 (\mathrm{mod}\, 2)$.
6) Repeat $n-1$ times to get $n-1$ linearly independent equations for s. Solve the system to find s.

8. Advanced Topics

8.1. Density Matrices (Blīvuma matricas ρ)

- Describes quantum states, including mixed states (statistical ensemble of pure states).
- Pure state $|\psi\rangle$: $\rho = |\psi\rangle\langle\psi|$.
- Mixed state: $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where p_i are probabilities, $\sum p_i = 1$.
- Properties:
  - $Tr(\rho) = 1$.
  - $\rho^\dagger = \rho$ (Hermitian).
  - ρ is positive semi-definite (eigenvalues ≥ 0).
- Evolution: $\rho' = U\rho U^\dagger$.
- Measurement: Probability of outcome m : $P(m) = Tr\big(M_m^\dagger M_m \rho\big)$. Post-measurement state: $\frac{M_m \rho M_m^\dagger}{Tr(M_m \rho M_m^\dagger)}$
- Purity: $Tr(\rho)^2 \leq 1$. $Tr(\rho)^2 = 1$ if ρ is a pure state.
- Partial Trace (Tr_B): If ρ_{AB} describes system AB, $\rho_A = Tr_{B(\rho_{AB})}$ describes system A.

8.2. Quantum Cryptography

8.2.1. BB84 Protocol

1) Alice chooses random bits and random bases (rectilinear $+$ or diagonal $\times$) for each bit.
   - $0 \xrightarrow{+} |0\rangle, 1 \xrightarrow{+} |1\rangle$
   - $0 \xrightarrow{\times} |+\rangle, 1 \xrightarrow{\times} |-\rangle$
2) Alice sends qubits to Bob.
3) Bob chooses random bases to measure each qubit.
4) Alice and Bob publicly announce their basis choices. They keep bits where bases matched (sifted key).
5) They sacrifice a portion of the sifted key to estimate error rate (detect eavesdropping). If error rate is low, remaining bits form the secret key.

8.2.2. Security

Eavesdropping (Eve) introduces errors because she doesn't know Alice's bases and her measurements disturb the states.

8.3. Quantum Error Correction

Protects quantum states from decoherence and errors.

8.3.1. 3-Qubit Bit Flip Code

- Encoding: $|0\rangle \to |0_L\rangle = |000\rangle, |1\rangle \to |1_L\rangle = |11\rangle$.
- Error detection: Measure stabilizers Z_1Z_2, Z_2Z_3.
- Correction: If Z_1Z_2 flips, error on $Q1$ or $Q2$. If Z_2Z_3 flips, error on $Q2$ or $Q3$. (e.g., if $Z_1Z_2 = -1$, $Z_2Z_3 = +1 \implies$ error on $Q1$, apply X_1).

8.3.2. 3-Qubit Phase Flip Code

- Encoding: $|0\rangle \to |+_L\rangle = |{+}{+}{+}\rangle$, $|1\rangle \to |-_L\rangle = |{-}{-}{-}\rangle$. (Hadamard basis of bit flip code).
- Error detection: Measure stabilizers X_1X_2, X_2X_3.

8.3.3. Shor's 9-Qubit Code

Corrects arbitrary single-qubit errors (bit flips, phase flips, or both). Concatenates bit-flip and phase-flip codes.

$$|0\rangle \to \frac{1}{2\sqrt{2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$$

$$|1\rangle \to \frac{1}{2\sqrt{2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$$

- Stabilizer Codes: A general framework for QEC. Code space is the simultaneous $+1$ eigenspace of a set of commuting Pauli operators (stabilizers).